

Test 6

Section I, Part A

Allotted Time: 62 Minutes

Number of Questions: 29 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

Instructions: Work through each of the problems, using the space provided to jot down calculations or notes. Carefully examine the answer choices for each question and determine which is the most accurate. Once you've selected the best answer, fill in the matching oval on the answer sheet. Keep in mind that anything written in this booklet won't be graded. Don't get stuck on any single question for too long. In this test, unless stated otherwise, assume that the domain of a function f includes all real values of x that result in $f(x)$ being a real number:

1. Evaluate the following limit:

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$$

- a. 4
 - b. 8
 - c. 12
 - d. The limit does not exist
2. If

$$f(x) = 3x^5 - 4e^x + 7 \sin x,$$

what is $f'(x)$?

- a. $15x^4 - 4e^x + 7 \cos x$
- b. $15x^4 - 4e^x - 7 \cos x$
- c. $3x^4 - 4e^x + 7 \cos x$
- d. $15x^4 + 4e^x + 7 \cos x$

3. Let

$$g(x) = \sqrt{2x^3 + 1}.$$

Find $g'(x)$.

- a. $\frac{3x^2}{\sqrt{2x^3+1}}$
- b. $\frac{6x^2}{\sqrt{2x^3+1}}$
- c. $3x^2\sqrt{2x^3+1}$
- d. $\frac{1}{2\sqrt{2x^3+1}}$

4. Suppose

$$f'(x) = (x + 3)^2(x - 2).$$

Which statement is true?

- a. f has local minima at $x = -3$ and $x = 2$.
- b. f has a local maximum at $x = -3$ and a local minimum at $x = 2$.
- c. f has no local extremum at $x = -3$ and a local minimum at $x = 2$.
- d. f has a local minimum at $x = -3$ and no local extremum at $x = 2$.

5. Evaluate

$$\int_1^3 (2x + 5) \, dx.$$

- a. 10
- b. 14
- c. 18
- d. 22

6. Let

$$f(x) = \begin{cases} kx^2 + 1, & x < 2 \\ 3x + 3, & x \geq 2 \end{cases}$$

What value of k makes f continuous at $x = 2$?

- a. 2
- b. $\frac{5}{2}$
- c. 3
- d. 4

7. The position of a particle moving along a line is given by

$$s(t) = t^3 - 4t^2 + 6t$$

for $t \geq 0$. What is the acceleration of the particle at $t = 3$?

- a. 2
- b. 4
- c. 6
- d. 10

8. If

$$h(x) = x^3 \cos x,$$

then $h'(x)$ is

- a. $3x^2 \sin x + x^3 \cos x$
- b. $3x^2 \cos x + x^3 \sin x$
- c. $3x^2 \cos x - x^3 \sin x$
- d. $x^3 \cos x - 3x^2 \sin x$

9. Suppose

$$f''(x) = (x + 1)(x - 4).$$

Which statement is true?

- a. f is concave up on $(-1, 4)$ and concave down on $(-\infty, -1) \cup (4, \infty)$.
- b. f is concave up on $(-\infty, -1) \cup (4, \infty)$ and concave down on $(-1, 4)$.
- c. f is concave down on $(-\infty, 4)$ and concave up on $(4, \infty)$.
- d. f is concave up on $(-\infty, -1)$ only.

10. Find the area of the region bounded by the graph of

$$y = x^2 - 4$$

and the x -axis on the interval $[-3, 3]$.

- a. $\frac{20}{3}$
- b. $\frac{28}{3}$
- c. $\frac{46}{3}$
- d. $\frac{52}{3}$

11. The graph of f consists of line segments from $(0, 2)$ to $(3, 2)$, from $(3, 2)$ to $(5, -2)$, and from $(5, -2)$ to $(7, 0)$. What is

$$\int_0^7 f(x) \, dx?$$

- a. 4
- b. 6
- c. 8
- d. 10

12. The curve is defined by

$$x^2y + y^2 = 12.$$

What is $\frac{dy}{dx}$ at the point $(2, 2)$?

- a. 1
- b. $\frac{1}{2}$
- c. $-\frac{1}{2}$
- d. -1

13. Let

$$f(x) = x^2 - 4x$$

on the interval $[0, 4]$. What value of c satisfies the conclusion of Rolle's Theorem?

- a. 2
- b. 1
- c. 4
- d. 3

14. Evaluate the right-hand limit

$$\lim_{x \rightarrow 2^+} \left(\frac{1}{(x-2)^2} \right).$$

- a. $-\infty$
- b. ∞
- c. 0
- d. The limit does not exist

15. Let

$$G(x) = \int_0^{\sin x} t^2 dt.$$

What is $G'(x)$?

- a. $\sin^2(x) \cos(x)$
- b. $2 \sin(x) \cos(x)$
- c. $\sin^2(x)$
- d. $\cos^2(x) \sin(x)$

16. A square has side length s , and the side length is increasing at a rate of 3 centimeters per second. At the instant when $s = 5$ centimeters, how fast is the area of the square increasing?

- a. $30 \text{ cm}^2/\text{s}$
- b. $15 \text{ cm}^2/\text{s}$
- c. $45 \text{ cm}^2/\text{s}$
- d. $75 \text{ cm}^2/\text{s}$

17. For the differential equation $\frac{dy}{dx} = y(3-x)$, at which point is the slope of the solution curve equal to 0?

- a. (1, 2)
- b. (2, 3)
- c. (3, 5)
- d. (4, 1)

18. If

$$h(x) = \frac{x^2 + 1}{e^x},$$

then $h'(x)$ is

- a. $\frac{2x}{e^x}$
- b. $\frac{x^2+2x+1}{e^x}$
- c. $\frac{x^2-2x+1}{e^x}$
- d. $\frac{2x-x^2-1}{e^x}$

19. Evaluate

$$\int_0^2 x \sqrt{x^2 + 5} \, dx.$$

- a. $\frac{27-5\sqrt{5}}{3}$
- b. $27 - 5\sqrt{5}$
- c. $\frac{9-\sqrt{5}}{3}$
- d. $\frac{27+5\sqrt{5}}{3}$

20. Let

$$f(x) = x^3 - 6x^2 + 9x + 2$$

on the interval $[0, 5]$. What is the absolute maximum value of f on the interval?

- a. 22
- b. 6
- c. 2
- d. 5

21. Suppose $f(3) = 7$ and $f'(3) = 2$. What is $(f^{-1})'(7)$?

- a. 2
- b. -2
- c. $\frac{1}{2}$
- d. $-\frac{1}{2}$

22. Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{2x}.$$

- a. 0
- b. 1
- c. $\frac{2}{5}$
- d. $\frac{5}{2}$

23. Which definite integral is represented by

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 + \frac{3i}{n}\right)^2 \cdot \frac{3}{n}?$$

- a. $\int_2^5 x^2 dx$
- b. $\int_0^3 (2+x)^2 dx$
- c. $\int_0^2 (3+x)^2 dx$
- d. $\int_2^3 x^2 dx$

24. The graph of $f'(x)$ has the following sign behavior:

- $f'(x) < 0$ for $x < -2$
- $f'(-2) = 0$
- $f'(x) > 0$ for $-2 < x < 1$
- $f'(1) = 0$
- $f'(x) > 0$ for $1 < x < 4$
- $f'(4) = 0$
- $f'(x) < 0$ for $x > 4$

Which statement is true?

- a. f has local maxima at $x = -2$ and $x = 4$.
- b. f has local minima at $x = -2$ and $x = 1$.
- c. f has no local extrema.
- d. f has a local minimum at $x = -2$, no local extremum at $x = 1$, and a local maximum at $x = 4$.

25. A particle moves along a line with velocity

$$v(t) = t^2 - 7t + 10$$

and acceleration

$$a(t) = 2t - 7$$

for $0 < t < 7$. On which interval(s) is the particle speeding up?

- a. $\left(2, \frac{7}{2}\right)$ and $(5, 7)$
- b. $(0, 2)$ and $(5, 7)$
- c. $\left(\frac{7}{2}, 5\right)$ only
- d. $(2, 5)$ only

26. Which function satisfies

$$\frac{dy}{dx} = 3x^2y$$

and the initial condition $y(0) = 2$?

- a. $y = e^{3x^2} + 2$
- b. $y = 2e^{3x^2}$
- c. $y = e^{x^3} + 2$
- d. $y = 2e^{x^3}$

27. The region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ is revolved about the x -axis. What is the volume of the resulting solid?

- a. 4π
- b. 8π
- c. 16π
- d. 32π

28. Evaluate

$$\lim_{h \rightarrow 0} \frac{(2+h)^4 - 16}{h}.$$

- a. 8
- b. 16
- c. 32
- d. 64

29. A rectangular garden is built against a straight wall, so fencing is needed for only three sides. If 60 meters of fencing are available, what is the maximum possible area of the garden?
- a. 450 square meters
 - b. 400 square meters
 - c. 300 square meters
 - d. 900 square meters

Section I, Part B

Allotted Time: 38 Minutes

Number of Questions: 13 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: For each problem below, solve using the space provided for scratch work. Review the format of the answer options, select the best choice, and mark the corresponding oval on your answer sheet. Note that any work written in the test booklet will not receive credit, so focus on completing the answer sheet. Avoid spending excessive time on any single question. The exact numerical answer may not always be included in the answer choices. In such cases, select the option that most closely approximates the exact value. Unless stated otherwise, the domain of a function is assumed to include all real numbers x for which $f(x)$ produces a real number.

30. Values of a continuous function f are shown below.

x	0	1	3	4.5	6	8
$f(x)$	2.4	3.1	4.7	4.2	3.5	2.8

Using the trapezoidal rule with the intervals shown in the table, approximate $\int_0^8 f(x) \, dx$.

- a. 25.6
- b. 27.8
- c. 29.3
- d. 31.5

31. The curve is defined implicitly by

$$x^2 + 2xy + 4y^2 = 20.$$

At the point $(1.8, 2.1)$, approximate $\frac{dy}{dx}$.

- a. -0.72
- b. -0.38
- c. 0.38
- d. 0.72

32. Let $g(x) = \cos(1.5x) + \ln(x + 2) - 0.3x$ on $[0, 5]$. Use a calculator to determine the absolute maximum value of g on $[0, 5]$.

- a. 0.79
- b. 1.15
- c. 1.70
- d. 2.34

33. A particle's position $s(t)$, in meters, is shown in the table below:

t	2.8	2.9	3.0	3.1	3.2
$s(t)$	14.21	15.36	16.56	17.84	19.17

What is the best estimate of $s'(3)$.

- a. 11.2
- b. 12.4
- c. 13.1
- d. 14.8

34. Values of a function $h(x)$ near $x = 2$ are shown.

x	1.99	1.999	1.9999	2.0001	2.001	2.01
$h(x)$	4.91	4.99	5.00	5.00	5.01	5.09

Based on the table, which statement is most accurate?

- a. $\lim_{x \rightarrow 2} h(x) = 4$
- b. $\lim_{x \rightarrow 2} h(x) = 5$
- c. $\lim_{x \rightarrow 2} h(x)$ does not exist
- d. $h(2) = 5$

35. Let

$$A(x) = \int_0^x (e^{-t^2} + \sqrt{t+1}) dt.$$

Use a calculator to approximate $A(2)$.

- a. 2.18
- b. 2.94
- c. 3.21
- d. 3.68

36. The region enclosed by $y = \ln(x+1)$ and $y = 0.4x$ for $x \geq 0$ is bounded by their intersection at $x = 0$ and one additional positive intersection. Use a calculator to find the area of the region.

- a. 0.85
- b. 1.27
- c. 1.91
- d. 2.48

37. A function y satisfies

$$\frac{dy}{dx} = \frac{x^2}{y+2}, \quad y(0) = 0.$$

Find $y(3)$. Round to the nearest hundredth.

- a. 1.58
- b. 2.12
- c. 2.41
- d. 2.69

38. Let

$$f(x) = e^{-x} \cos(x^2).$$

Use a calculator to approximate $f'(1.2)$.

- a. 0.31
- b. -0.31
- c. 0.76
- d. -0.76

39. A twice-differentiable function f has

$$f''(x) = \cos x - 0.2x.$$

Use a calculator to approximate the x -value where f has a point of inflection on $[0, 4]$.

- a. 0.64
- b. 0.92
- c. 1.08
- d. 1.31

40. The number of units produced by a machine, in hundreds, after t hours is modeled by

$$P(t) = 80(1 - e^{-0.2t}) - 3t$$

for $0 \leq t \leq 20$. Use a calculator to determine when production is greatest.

- a. 5.42 hours
- b. 6.91 hours
- c. 8.37 hours
- d. 11.06 hours

41. Find the average value of

$$f(x) = \sqrt{4 + x^3}$$

on the interval $[1, 3]$.

- a. 2.84
- b. 3.61
- c. 4.28
- d. 5.02

42. The region bounded by

$$y = e^{-x}, \quad y = \frac{x}{2}, \quad x = 0$$

is revolved about the x -axis. Use a calculator to approximate the volume of the resulting solid.

- a. 0.74
- b. 1.12
- c. 1.86
- d. 2.35

Section II, Part A

Allotted Time: 30 Minutes

Number of Questions: 2 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: Providing your set up for these problems is essential, as partial credit may be granted. If you include decimal approximations, ensure they are accurate to three decimal places.

1. A greenhouse uses a heating system to control the temperature inside the building. The temperature inside the greenhouse, in degrees Fahrenheit, t hours after midnight is modeled by

$$T(t) = 68 + 9te^{-0.18t} + 4\sin(0.7t)$$

for $0 \leq t \leq 12$.

- Find $T'(t)$. Include units.
 - Use a calculator to find all critical values of T on the interval $0 < t < 12$. Give your answers to three decimal places.
 - Find the absolute maximum temperature inside the greenhouse on the interval $0 \leq t \leq 12$. Include units and justify your answer.
 - The temperature is increasing at a decreasing rate when $T'(t) > 0$ and $T''(t) < 0$. Determine the interval(s) on $0 < t < 12$ where the temperature is increasing at a decreasing rate. Justify your answer.
2. At a trailhead parking lot, cars enter the lot at a rate of $E(t)$ cars per hour, where t is measured in hours after 6:00 a.m. Selected values of $E(t)$ are shown in the table below.

t	0	1	2	3.5	5	7	8
$E(t)$	18	26	35	42	31	24	16

At time $t = 0$, there are 35 cars in the lot. Cars leave the lot at a rate modeled by

$$L(t) = 4 + 2\sqrt{t+1}$$

cars per hour, for $0 \leq t \leq 8$.

- Use the trapezoidal rule with the data in the table to approximate

$$\int_0^8 E(t) dt.$$

- b. Interpret the meaning of

$$\int_0^8 E(t) dt$$

in the context of the problem. Include units.

- c. Find the average rate at which cars enter the lot over the interval $0 \leq t \leq 8$. Include units.
- d. Use your answer from part (a) and the model for $L(t)$ to approximate the number of cars in the lot at time $t = 8$.

Section II, Part B

Allotted Time: 60 Minutes

Number of Questions: 4 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

3. Let f be the piecewise function

$$f(x) = \begin{cases} ae^x + bx, & x \leq 0 \\ \ln(x+1) + cx + d, & x > 0 \end{cases}$$

where a , b , c , and d are constants.

- Find the value of d in terms of a so that f is continuous at $x = 0$.
- Find a relationship among a , b , and c so that f is differentiable at $x = 0$.
- Assume f is continuous and differentiable at $x = 0$. Write an equation of the tangent line to the graph of f at $x = 0$.
- Suppose $a = 3$ and $c = 5$. Find the values of b and d that make f continuous and differentiable at $x = 0$.

4. Let

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2 + 3.$$

- Find $f'(x)$ and determine all critical values of f .
- On which intervals is f increasing? On which intervals is f decreasing? Justify your answer.
- Find all x -values where f has a point of inflection. Justify your answer.
- Verify that the hypotheses of the Mean Value Theorem are satisfied for f on the interval $[0, 4]$. Then find all values of c in $(0, 4)$ that satisfy the conclusion of the Mean Value Theorem.

5. Let f be a continuous function defined on $[0, 9]$. The graph of f is made of the following pieces:
- From $x = 0$ to $x = 2$, f is a line segment from $(0, -2)$ to $(2, 2)$.
 - From $x = 2$ to $x = 6$, f is the upper semicircle with center $(4, 2)$ and radius 2.
 - From $x = 6$ to $x = 9$, f is the line segment from $(6, 2)$ to $(9, -1)$.

Define

$$g(x) = 5 + \int_0^x f(t) dt.$$

- Find $g(2)$.
 - Find $g'(7)$.
 - On which interval(s) in $[0, 9]$ is g increasing? Justify your answer.
 - Find the absolute maximum value and absolute minimum value of $g(x)$ on $[0, 7]$. Justify your answer.
6. A function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{x+1}{y+2}$$

with initial condition

$$f(0) = 1.$$

- Show that $y = -2 + \sqrt{x^2 + 2x + 9}$ is a solution to the differential equation with the given initial condition.
- Find the general solution to the differential equation.
- Find the particular solution that satisfies $f(0) = 1$ and determine $f(3)$.
- For the particular solution from part (c), determine the interval(s) of x for which the solution is increasing and decreasing. Justify your answer.

Solutions

Section I, Part A

1. B	2. A	3. A	4. C	5. C
6. A	7. D	8. C	9. B	10. C
11. A	12. D	13. A	14. B	15. A
16. A	17. C	18. D	19. A	20. A
21. C	22. D	23. A	24. D	25. A
26. D	27. B	28. C	29. A	

1. Factor the numerator:

$$x^2 - 16 = (x - 4)(x + 4)$$

For $x \neq 4$,

$$\frac{x^2 - 16}{x - 4} = \frac{(x - 4)(x + 4)}{x - 4} = x + 4.$$

So,

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = 4 + 4 = 8.$$

2. Differentiate term-by-term:

$$\frac{d}{dx}(3x^5) = 15x^4, \quad \frac{d}{dx}(-4e^x) = -4e^x, \quad \frac{d}{dx}(7 \sin x) = 7 \cos x.$$

Therefore,

$$f'(x) = 15x^4 - 4e^x + 7 \cos x.$$

3. Rewrite the function as

$$g(x) = (2x^3 + 1)^{1/2}.$$

Using the chain rule,

$$g'(x) = \frac{1}{2}(2x^3 + 1)^{-1/2} (6x^2).$$

So,

$$g'(x) = \frac{3x^2}{\sqrt{2x^3 + 1}}.$$

4. The factor $(x + 3)^2$ does not change sign at $x = -3$, so $f'(x)$ does not change sign there. Therefore, f has no local extremum at $x = -3$. The sign of $f'(x)$ changes from negative to positive at $x = 2$, so f has a local minimum at $x = 2$.
5. Find an antiderivative:

$$\int (2x + 5) dx = x^2 + 5x.$$

Then

$$\int_1^3 (2x + 5) dx = [x^2 + 5x] \Big|_1^3 = (9 + 15) - (1 + 5) = 18.$$

6. Continuity at $x = 2$ requires

$$k(2)^2 + 1 = 3(2) + 3.$$

So,

$$4k + 1 = 9,$$

which gives

$$k = 2.$$

7. Velocity is the derivative of position:

$$v(t) = s'(t) = 3t^2 - 8t + 6.$$

Acceleration is

$$a(t) = v'(t) = s''(t) = 6t - 8.$$

At $t = 3$,

$$a(3) = 6(3) - 8 = 10.$$

8. Use the product rule:

$$h'(x) = 3x^2 \cos x + x^3 (-\sin x).$$

Therefore,

$$h'(x) = 3x^2 \cos x - x^3 \sin x.$$

9. Concavity is determined by the sign of $f''(x)$. Since

$$f''(x) = (x + 1)(x - 4),$$

the zeros are $x = -1$ and $x = 4$. Testing intervals shows $f''(x) > 0$ on $(-\infty, -1) \cup (4, \infty)$ and $f''(x) < 0$ on $(-1, 4)$. Therefore, f is concave up on $(-\infty, -1) \cup (4, \infty)$ and concave down on $(-1, 4)$.

10. The function crosses the x -axis at $x = -2$ and $x = 2$. The total area is

$$2 \int_2^3 (x^2 - 4) \, dx + \int_{-2}^2 (4 - x^2) \, dx.$$

First,

$$\int_2^3 (x^2 - 4) \, dx = \left[\frac{x^3}{3} - 4x \right]_2^3 = \frac{7}{3}.$$

So the two outside regions have total area

$$2 \cdot \frac{7}{3} = \frac{14}{3}.$$

Next,

$$\int_{-2}^2 (4 - x^2) \, dx = \frac{32}{3}.$$

Therefore, the total area is

$$\frac{14}{3} + \frac{32}{3} = \frac{46}{3}.$$

11. The integral is the signed area under the graph. From 0 to 3, the area is a rectangle:

$$3 \cdot 2 = 6.$$

From 3 to 5, the positive and negative triangular areas cancel, so the signed area is 0. From 5 to 7, the graph forms a triangle below the x -axis:

$$-\frac{1}{2}(2)(2) = -2.$$

Therefore,

$$\int_0^7 f(x) dx = 6 + 0 - 2 = 4.$$

12. Differentiate implicitly:

$$x^2y + y^2 = 12$$

$$2xy + x^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 0.$$

Factor out $\frac{dy}{dx}$:

$$(x^2 + 2y) \frac{dy}{dx} = -2xy.$$

So

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y}.$$

At $(2, 2)$,

$$\frac{dy}{dx} = \frac{-2(2)(2)}{2^2 + 2(2)} = \frac{-8}{8} = -1.$$

13. Since f is a polynomial, it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Also,

$$f(0) = 0$$

and

$$f(4) = 4^2 - 4(4) = 16 - 16 = 0.$$

Therefore, Rolle's Theorem applies. Find c such that $f'(c) = 0$.

$$f'(x) = 2x - 4.$$

Set $f'(c) = 0$:

$$2c - 4 = 0,$$

so

$$c = 2.$$

14. As x approaches 2 from the right, the quantity $(x - 2)$ is positive and approaches 0. Therefore, $(x - 2)^2$ is positive and approaches 0. So

$$\frac{1}{(x - 2)^2} \rightarrow \infty.$$

15. Use the Fundamental Theorem of Calculus and the chain rule. Since the upper limit is $\sin x$,

$$G'(x) = (\sin x)^2 \cdot \frac{d}{dx}(\sin x).$$

Therefore,

$$G'(x) = \sin^2(x) \cos(x).$$

16. The area of a square is

$$A = s^2.$$

Differentiate with respect to time:

$$\frac{dA}{dt} = 2s \frac{ds}{dt}.$$

At $s = 5$ and $\frac{ds}{dt} = 3$,

$$\frac{dA}{dt} = 2(5)(3) = 30.$$

17. The slope is zero when

$$\frac{dy}{dx} = 0.$$

For this differential equation,

$$y(3 - x) = 0.$$

So, the slope is zero when $y = 0$ or $x = 3$. The only listed point that satisfies this is $(3, 5)$.

18. Rewrite the function as

$$h(x) = (x^2 + 1)e^{-x}.$$

Use the product rule:

$$h'(x) = 2xe^{-x} + (x^2 + 1)(-e^{-x}).$$

Factor out e^{-x} :

$$h'(x) = e^{-x}(2x - x^2 - 1).$$

This is equivalent to

$$h'(x) = \frac{2x - x^2 - 1}{e^x}.$$

19. Let

$$u = x^2 + 5.$$

Then

$$du = 2x \, dx,$$

so

$$x \, dx = \frac{1}{2} du.$$

When $x = 0$, $u = 5$. When $x = 2$, $u = 9$. Therefore,

$$\int_0^2 x \sqrt{x^2 + 5} \, dx = \frac{1}{2} \int_5^9 u^{1/2} \, du.$$

Integrate:

$$\frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_5^9 = \frac{1}{3} (9^{3/2} - 5^{3/2}).$$

Since $9^{3/2} = 27$ and $5^{3/2} = 5\sqrt{5}$,

$$\int_0^2 x \sqrt{x^2 + 5} \, dx = \frac{27 - 5\sqrt{5}}{3}.$$

20. First find the derivative:

$$f'(x) = 3x^2 - 12x + 9.$$

Factor:

$$f'(x) = 3(x - 1)(x - 3).$$

The critical values are $x = 1$ and $x = 3$. Check these values and the endpoints of $[0, 5]$:

$$f(0) = 2,$$

$$f(1) = 1 - 6 + 9 + 2 = 6,$$

$$f(3) = 27 - 54 + 27 + 2 = 2,$$

$$f(5) = 125 - 150 + 45 + 2 = 22.$$

The absolute maximum value is 22.

21. Applying the derivative of an inverse formula:

$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(3)} = \frac{1}{2}$$

22. Rewrite the limit as

$$\frac{\sin(5x)}{2x} = \frac{5}{2} \cdot \frac{\sin(5x)}{5x}.$$

Since

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1,$$

the limit is

$$\frac{5}{2}.$$

23. The factor outside the function value is

$$\Delta x = \frac{3}{n},$$

so the interval has length 3. The sample point is

$$x_i = 2 + \frac{3i}{n},$$

which starts at 2 and ends at 5. Therefore, the Riemann sum represents

$$\int_2^5 x^2 \, dx.$$

24. At $x = -2$, $f'(x)$ changes from negative to positive, so f has a local minimum. At $x = 1$, $f'(x)$ does not change sign, so there is no local extremum. At $x = 4$, $f'(x)$ changes from positive to negative, so f has a local maximum.

25. A particle is speeding up when velocity and acceleration have the same sign. Since

$$v(t) = t^2 - 7t + 10 = (t - 2)(t - 5),$$

$v(t) > 0$ on $(0, 2)$ and $(5, 7)$, and $v(t) < 0$ on $(2, 5)$. Also,

$$a(t) = 2t - 7,$$

so $a(t) < 0$ on $(0, \frac{7}{2})$ and $a(t) > 0$ on $(\frac{7}{2}, 7)$. The signs match on

$$(2, \frac{7}{2}) \text{ and } (5, 7).$$

26. Separate variables:

$$\frac{1}{y} dy = 3x^2 dx.$$

Integrate both sides:

$$\ln|y| = x^3 + C.$$

Therefore,

$$y = Ce^{x^3}.$$

Using $y(0) = 2$ gives $C = 2$, so

$$y = 2e^{x^3}.$$

27. Using disks, the radius of each cross section is $\sqrt{x}$. Therefore,

$$V = \pi \int_0^4 (\sqrt{x})^2 dx.$$

So

$$V = \pi \int_0^4 x dx = \pi \left[\frac{x^2}{2} \right]_0^4 = 8\pi.$$

28. The limit matches the definition of the derivative of

$$f(x) = x^4$$

at $x = 2$. Since

$$f'(x) = 4x^3,$$

we get

$$f'(2) = 4(2)^3 = 32.$$

29. Let x be the width perpendicular to the wall and y be the side parallel to the wall. Since fencing is needed for three sides,

$$2x + y = 60.$$

Therefore,

$$y = 60 - 2x.$$

The area is

$$A = xy = x(60 - 2x) = 60x - 2x^2.$$

This quadratic opens downward. Its maximum occurs at

$$x = \frac{-b}{2a} = \frac{-60}{2(-2)} = 15.$$

Then

$$y = 60 - 2(15) = 30.$$

So the maximum area is

$$15 \cdot 30 = 450$$

square meters.

Section I, Part B

30. C	31. B	32. C	33. B	34. B
35. D	36. A	37. D	38. D	39. D
40. C	41. B	42. B		

30. Use trapezoids on each interval:

$$\frac{1}{2}(1)(2.4 + 3.1) + \frac{1}{2}(2)(3.1 + 4.7) + \frac{1}{2}(1.5)(4.7 + 4.2) + \frac{1}{2}(1.5)(4.2 + 3.5) + \frac{1}{2}(2)(3.5 + 2.8).$$

This gives

$$2.75 + 7.8 + 6.675 + 5.775 + 6.3 = 29.3.$$

31. Differentiate implicitly:

$$2x + 2(xy' + y) + 8yy' = 0.$$

Solve for y' :

$$y' = -\frac{2x + 2y}{2x + 8y} = -\frac{x + y}{x + 4y}.$$

At $(1.8, 2.1)$,

$$y' = -\frac{1.8 + 2.1}{1.8 + 4(2.1)} \approx -0.38.$$

32. Use the calculator's maximum feature on $[0, 5]$, or solve $g'(x) = 0$ and check endpoints. The absolute maximum value is approximately 1.70.

33. Calculate with symmetric difference quotient:

$$s'(3) \approx \frac{s(3.1) - s(2.9)}{3.1 - 2.9} = \frac{17.84 - 15.36}{0.2} = 12.4.$$

34. The values from both sides of $x = 2$ appear to approach 5, therefore

$$\lim_{x \rightarrow 2} h(x) = 5.$$

35. Use numerical integration:

$$A(2) = \int_0^2 (e^{-t^2} + \sqrt{t+1}) dt \approx 3.68.$$

36. Use a calculator to solve

$$\ln(x + 1) = 0.4x.$$

The positive intersection is 4.046970313. On $(0, 4.046970313)$, $\ln(x + 1) > 0.4x$, so the area is

$$\int_0^{4.046970313} (\ln(x + 1) - 0.4x) dx \approx 0.85.$$

37. Separate variables:

$$(y + 2) dy = x^2 dx.$$

Integrate:

$$\frac{y^2}{2} + 2y = \frac{x^3}{3} + C.$$

Since $y(0) = 0$, $C = 0$. At $x = 3$,

$$\frac{y^2}{2} + 2y = 9.$$

Solving gives

$$y \approx 2.69.$$

38. Use the numerical derivate feature on your calculator at $x = 1.2$. This gives $f'(1.2) \approx -0.76$.

39. Inflection points occur where $f''(x) = 0$ and concavity changes. Solving

$$\cos x - 0.2x = 0$$

on $[0, 4]$ gives

$$x \approx 1.31.$$

40. Use a calculator to maximize $P(t)$ on $[0, 20]$, or solve $P'(t) = 0$ and check endpoints. The maximum occurs at $t \approx 8.37$.

41. Average value is

$$\frac{1}{3-1} \int_1^3 \sqrt{4+x^3} dx.$$

Using calculator integration,

$$\frac{1}{2} \int_1^3 \sqrt{4+x^3} dx \approx 3.61.$$

42. First use a calculator to solve

$$e^{-x} = \frac{x}{2}.$$

The intersection is approximately $x = 0.852605502$. Using washers around the x -axis,

$$V = \pi \int_0^{0.852605502} \left(e^{-2x} - \left(\frac{x}{2} \right)^2 \right) dx.$$

Using calculator integration, $V \approx 1.12$.

Section II, Part A

1. (a) Differentiate:

$$T(t) = 68 + 9te^{-0.18t} + 4\sin(0.7t)$$

For the product $9te^{-0.18t}$, use the product rule:

$$\frac{d}{dt}(9te^{-0.18t}) = 9e^{-0.18t} + 9t(-0.18)e^{-0.18t}$$

So,

$$T'(t) = 9e^{-0.18t} - 1.62te^{-0.18t} + 2.8\cos(0.7t)$$

Equivalently,

$$T'(t) = 9e^{-0.18t}(1 - 0.18t) + 2.8\cos(0.7t)$$

The units are degrees Fahrenheit per hour.

- (b) Critical values occur where

$$T'(t) = 0$$

Using a calculator on $0 < t < 12$,

$$9e^{-0.18t}(1 - 0.18t) + 2.8\cos(0.7t) = 0$$

at approximately

$$t = 3.349, \quad t = 7.090, \quad t = 10.580$$

- (c) To find the absolute maximum on
- $[0, 12]$
- , check the endpoints and the critical values.

$$T(0) = 68$$

$$T(3.349) \approx 87.357$$

$$T(7.090) \approx 81.934$$

$$T(10.580) \approx 85.785$$

$$T(12) \approx 83.874$$

The greatest value is 87.357. Therefore, the absolute maximum temperature is approximately 87.357°F , and it occurs at $t \approx 3.349$ hours after midnight.

- (d) First find
- $T''(t)$
- .

From

$$T'(t) = 9e^{-0.18t}(1 - 0.18t) + 2.8\cos(0.7t),$$

we get

$$T''(t) = 9e^{-0.18t}(0.0324t - 0.36) - 1.96\sin(0.7t).$$

Using a calculator,

$$T''(t) = 0$$

at approximately

$$t = 5.024 \quad \text{and} \quad t = 8.880.$$

Testing signs:

- $T'(t) > 0$ on $(0, 3.349)$ and $(7.090, 10.580)$

- $T''(t) < 0$ on $(0, 5.024)$ and $(8.880, 12)$

The temperature is increasing at a decreasing rate where both conditions are true:

$$(0, 3.349) \cup (8.880, 10.580)$$

2. (a) Use trapezoids on each interval:

$$\begin{aligned} \int_0^8 E(t) dt &\approx \frac{1}{2}(1)(18 + 26) + \frac{1}{2}(1)(26 + 35) + \frac{1}{2}(1.5)(35 + 42) \\ &\quad + \frac{1}{2}(1.5)(42 + 31) + \frac{1}{2}(2)(31 + 24) + \frac{1}{2}(1)(24 + 16). \end{aligned}$$

Therefore,

$$\int_0^8 E(t) dt \approx 22 + 30.5 + 57.75 + 54.75 + 55 + 20 = 240.$$

About 240 cars enter the parking lot during the first 8 hours.

- (b) The integral

$$\int_0^8 E(t) dt$$

represents the total number of cars that enter the parking lot from 6:00 a.m. to 2:00 p.m. The units are cars.

- (c) The average rate is

$$\frac{1}{8 - 0} \int_0^8 E(t) dt.$$

Using the approximation from part (a),

$$\frac{240}{8} = 30.$$

The average rate at which cars enter the lot is approximately 30 cars per hour.

- (d) The number of cars in the lot at $t = 8$ is

$$35 + \int_0^8 E(t) dt - \int_0^8 L(t) dt.$$

From part (a),

$$\int_0^8 E(t) dt \approx 240.$$

Using a calculator,

$$\int_0^8 (4 + 2\sqrt{t+1}) dt \approx 66.667.$$

Therefore, the number of cars in the lot at $t = 8$ is approximately

$$35 + 240 - 66.667 = 208.333.$$

There are approximately 208 cars in the lot at 2:00 p.m.

Section II, Part B

3. (a) Continuity at $x = 0$ requires the left-hand value and right-hand limit to be equal.

From the left,

$$f(0) = ae^0 + b(0) = a.$$

From the right,

$$\ln(0 + 1) + c(0) + d = d.$$

Set these equal:

$$a = d.$$

Therefore,

$$d = a.$$

- (b) Differentiability at $x = 0$ requires the left and right derivatives to match.

For $x \leq 0$,

$$f'(x) = ae^x + b,$$

so the left-hand derivative at $x = 0$ is

$$a + b.$$

For $x > 0$,

$$f'(x) = \frac{1}{x+1} + c,$$

so the right-hand derivative at $x = 0$ is

$$1 + c.$$

Set the one-sided derivatives equal:

$$a + b = 1 + c.$$

Therefore,

$$a + b = c + 1.$$

- (c) If f is continuous and differentiable at $x = 0$, the point on the graph is

$$(0, f(0)) = (0, a).$$

The slope of the tangent line is

$$f'(0) = a + b.$$

Therefore, an equation of the tangent line is

$$y - a = (a + b)x.$$

Equivalent forms of this equation are also correct.

(d) From part (a), continuity requires

$$d = a.$$

Since $a = 3$,

$$d = 3.$$

From part (b), differentiability requires

$$a + b = c + 1.$$

Substitute $a = 3$ and $c = 5$:

$$3 + b = 5 + 1.$$

Thus,

$$b = 3.$$

Therefore,

$$b = 3 \text{ and } d = 3.$$

4. (a) First, find the derivative:

$$f'(x) = x^3 - 6x^2 + 8x.$$

Factor:

$$f'(x) = x(x^2 - 6x + 8) = x(x - 2)(x - 4).$$

Critical values occur where $f'(x) = 0$ or where $f'(x)$ is undefined. Since f' is defined for all real numbers,

$$x(x - 2)(x - 4) = 0.$$

Therefore, the critical values are

$$x = 0, \quad x = 2, \quad x = 4.$$

(b) Use the sign of $f'(x) = x(x - 2)(x - 4)$. This can be shown with a gradient diagram.

x	-1	0	1	2	3	4	5
Sign of $f'(x)$	-	0	+	0	-	0	+
Direction of $f(x)$							

Therefore, f is increasing on $(0, 2) \cup (4, \infty)$ and decreasing on $(-\infty, 0) \cup (2, 4)$.

(c) First, find the second derivative:

$$f''(x) = 3x^2 - 12x + 8.$$

Set $f''(x) = 0$:

$$3x^2 - 12x + 8 = 0.$$

Using the quadratic formula,

$$x = \frac{12 \pm \sqrt{144 - 96}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = 2 \pm \frac{2\sqrt{3}}{3}.$$

Since $f''(x)$ changes sign at both of these values, the graph of f has points of inflection at

$$x = 2 - \frac{2\sqrt{3}}{3} \quad \text{and} \quad x = 2 + \frac{2\sqrt{3}}{3}.$$

(d) Since f is a polynomial, it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Therefore, the Mean Value Theorem applies.

Compute the average rate of change:

$$\frac{f(4) - f(0)}{4 - 0}.$$

Evaluate the endpoints:

$$f(0) = 3$$

and

$$f(4) = \frac{1}{4}(4)^4 - 2(4)^3 + 4(4)^2 + 3 = 64 - 128 + 64 + 3 = 3.$$

So

$$\frac{f(4) - f(0)}{4} = \frac{3 - 3}{4} = 0.$$

Now set $f'(c) = 0$:

$$c(c - 2)(c - 4) = 0.$$

The solutions are $c = 0$, $c = 2$, and $c = 4$. The only value in the open interval $(0, 4)$ is

$$c = 2.$$

Therefore, $c = 2$.

5. (a) Since

$$g(2) = 5 + \int_0^2 f(t) dt,$$

we need the signed area from 0 to 2.

From $x = 0$ to $x = 2$, the graph is a line segment from $(0, -2)$ to $(2, 2)$. The signed area below the x -axis from 0 to 1 is a triangle with area 1, and the signed area above the x -axis from 1 to 2 is a triangle with area 1.

Thus,

$$\int_0^2 f(t) dt = -1 + 1 = 0.$$

Therefore,

$$g(2) = 5 + 0 = 5.$$

(b) By the Fundamental Theorem of Calculus,

$$g'(x) = f(x).$$

At $x = 7$, $f(7)$ lies on the line segment from $(6, 2)$ to $(9, -1)$. This line has slope

$$\frac{-1 - 2}{9 - 6} = -1.$$

Starting from $(6, 2)$, the value at $x = 7$ is

$$f(7) = 2 - 1 = 1.$$

Therefore,

$$g'(7) = 1.$$

(c) Since

$$g'(x) = f(x),$$

g is increasing where $f(x) > 0$.

On the first line segment, $f(x) = 0$ at $x = 1$. The semicircle from $x = 2$ to $x = 6$ is above the x -axis. On the last line segment, $f(x) = 0$ at $x = 8$.

Therefore, $f(x) > 0$ on

$$(1, 8).$$

So g is increasing on

$$(1, 8).$$

(d) The absolute extrema of g can occur at endpoints or where $g'(x) = 0$. Since $g'(x) = f(x)$, the candidates are

$$x = 0, \quad x = 1, \quad x = 8, \quad x = 9.$$

First,

$$g(0) = 5.$$

From 0 to 1, the graph is below the x -axis and forms a triangle with base 1 and height 2, so

$$g(1) = 5 - 1 = 4.$$

Next, find $g(8)$.

From 0 to 2, the signed area is 0. From 2 to 6, the area is a rectangle with width 4 and height 2, plus an upper semicircle of radius 2:

$$8 + \frac{1}{2}\pi(2)^2 = 8 + 2\pi.$$

From 6 to 8, the graph is above the x -axis and forms a triangle with base 2 and height 2, so the area is

$$\frac{1}{2}(2)(2) = 2.$$

Therefore,

$$g(8) = 5 + 0 + (8 + 2\pi) + 2 = 15 + 2\pi.$$

Finally, from 8 to 9, the graph is below the x -axis and forms a triangle with base 1 and height 1, so

$$g(9) = 15 + 2\pi - \frac{1}{2} = \frac{29}{2} + 2\pi.$$

Comparing the values,

$$g(0) = 5, \quad g(1) = 4, \quad g(8) = 15 + 2\pi, \quad g(9) = \frac{29}{2} + 2\pi.$$

The absolute minimum value is 4, and the absolute maximum value is $15 + 2\pi$.

6. (a) Given

$$y = -2 + \sqrt{x^2 + 2x + 9},$$

we have

$$y + 2 = \sqrt{x^2 + 2x + 9}.$$

Differentiate:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2 + 2x + 9}}(2x + 2) = \frac{x + 1}{\sqrt{x^2 + 2x + 9}}$$

Since

$$y + 2 = \sqrt{x^2 + 2x + 9},$$

this becomes

$$\frac{dy}{dx} = \frac{x + 1}{y + 2}.$$

So, the function satisfies the differential equation.

Also,

$$y(0) = -2 + \sqrt{0^2 + 2(0) + 9} = -2 + 3 = 1.$$

Therefore, the function satisfies the given initial condition.

(b) Separate the variables:

$$\frac{dy}{dx} = \frac{x + 1}{y + 2}$$

$$(y + 2) dy = (x + 1) dx.$$

Integrate both sides:

$$\int (y + 2) dy = \int (x + 1) dx.$$

$$\frac{y^2}{2} + 2y = \frac{x^2}{2} + x + C.$$

An equivalent form is

$$(y + 2)^2 = x^2 + 2x + C.$$

(c) Use the initial condition $f(0) = 1$ in

$$(y + 2)^2 = x^2 + 2x + C.$$

Substitute $x = 0$ and $y = 1$:

$$(1 + 2)^2 = 0^2 + 2(0) + C.$$

Thus,

$$9 = C.$$

So the particular solution is

$$(y + 2)^2 = x^2 + 2x + 9.$$

Since $f(0) = 1$, we choose the positive square root branch:

$$y = -2 + \sqrt{x^2 + 2x + 9}.$$

Now find $f(3)$:

$$f(3) = -2 + \sqrt{3^2 + 2(3) + 9} = -2 + \sqrt{24}.$$

Therefore,

$$f(3) = -2 + 2\sqrt{6}.$$

(d) The solution is increasing where $\frac{dy}{dx} > 0$.

Using the differential equation:

$$f'(x) = \frac{x + 1}{y + 2}.$$

For the particular solution,

$$y + 2 = \sqrt{x^2 + 2x + 9}.$$

Since

$$\sqrt{x^2 + 2x + 9} > 0$$

for all real x , the sign of $f'(x)$ depends only on the sign of $x + 1$.

Therefore,

$$f'(x) < 0 \text{ when } x < -1$$

and

$$f'(x) > 0 \text{ when } x > -1.$$

So the particular solution is decreasing on

$$(-\infty, -1)$$

and increasing on

$$(-1, \infty).$$